MATH 253

1 Vectors and geometry in R^2 and R^3

Normalizing

$\hat{V} = \dfrac{\vec{V}}{|V|}$

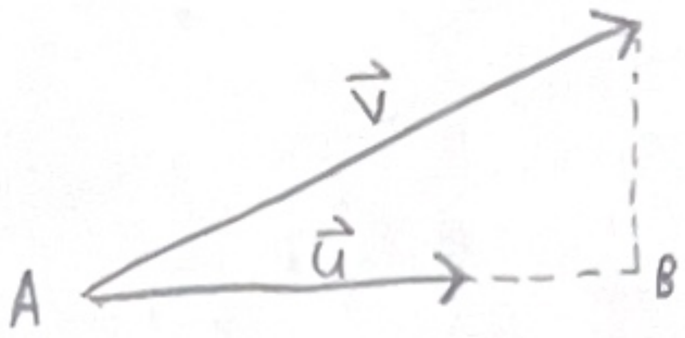
Standard Position

- a vector with its tail placed at the origin

Projections

$\text{proj}_{\vec{u}}\vec{V} = \left(\dfrac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$

"V onto u"



$\text{proj}_{\vec{u}}\vec{V} = \vec{AB}$

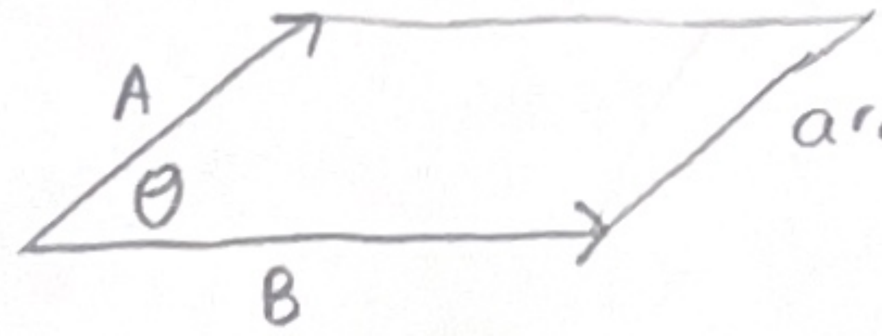
Cross Product

$|\vec{u}\times\vec{v}| = |u||v|\sin\theta$

$\vec{u}\times\vec{v} = \det\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ U_x & U_y & U_z\\ V_x & V_y & V_z\end{vmatrix}$

$= (U_yV_z - U_zV_y)\hat{\imath} - (U_xV_z - U_zV_x)\hat{\jmath} + (U_xV_y - U_yV_x)\hat{k}$

area of parallelogram:

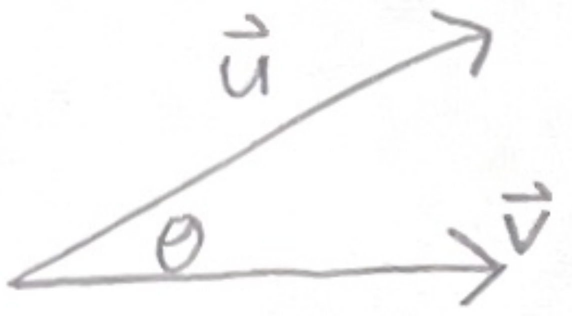


area $= |A||B|\sin\theta$

Dot Product

$\vec{u}\cdot\vec{v} = U_xV_x + U_yV_y + U_zV_z$

$\vec{u}\cdot\vec{v} = |u||v|\cos\theta$



Properties:

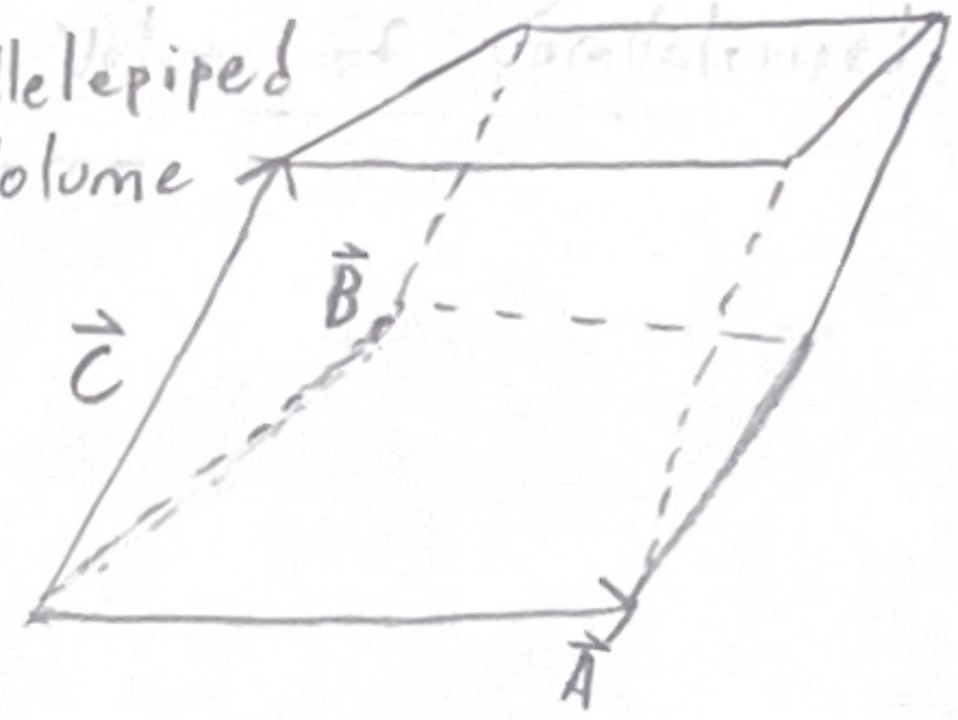
1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
2. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$
3. $K(\vec{u}\cdot\vec{v}) = (k\vec{v})\cdot\vec{v}$
4. $\vec{v}\cdot\vec{v} = |V|^2$

Symmetric form

$x = x_0 + ta$
$y = y_0 + tb$
$z = z_0 + tc$

$\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$

Parallelepiped Volume



Volume $= |(\vec{A}\times\vec{B})\cdot\vec{C}|$

Equation of a Line

$\langle x,y,z\rangle = (x_0,y_0,z_0) + t\langle A,B,C\rangle$

Parametric:

$x = x_0 + tA$
$y = y_0 + tB$
$z = z_0 + tC$

- where (x_0,y_0,z_0) is a point on the line and $\langle A,B,C\rangle$ is a direction vector parallel to the line

Equation of a Plane

$\langle x,y,z\rangle = (x_0,y_0,z_0) + t\langle U_1,U_2,U_3\rangle + S\langle V_1,V_2,V_3\rangle$

- where (x_0,y_0,z_0) is a point on the plane and $\vec{u},\vec{v}$ are vectors parallel to the plane and not parallel to each other

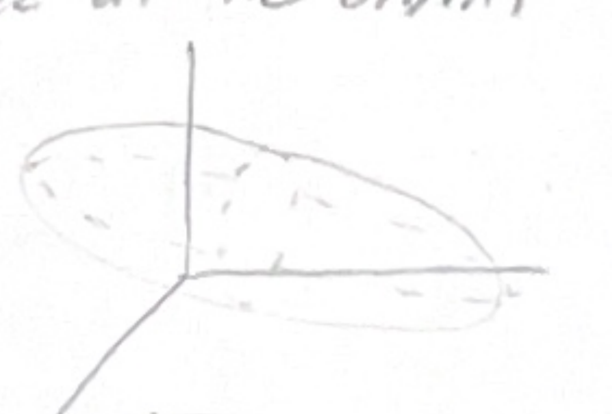
GENERAL Form: $Ax + By + Cz = D$

- where $\vec{n} = \langle A,B,C\rangle$ and you can plug in a point on a plane to solve for D

Quadric Surfaces and Functions of many Variables

Ellipsoid: (centered at the origin)

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

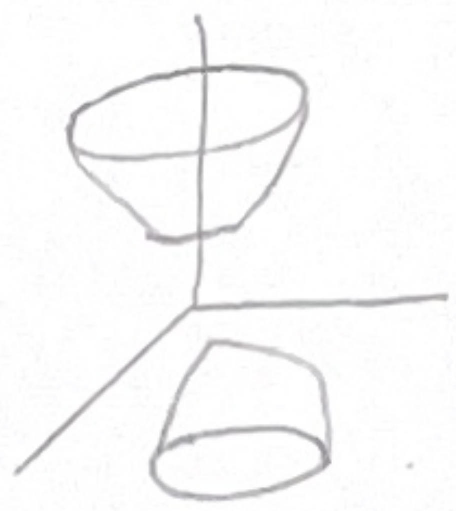


Cone:

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$

Hyperboloid of 2 sheets:

$-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

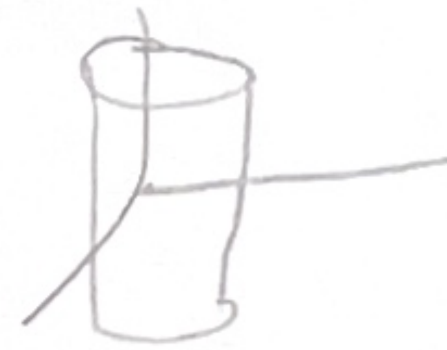


Hyperbolic Paraboloid: "Saddle"

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{z}{c}$

Cylinder:

$x^2 + y^2 = r^2$

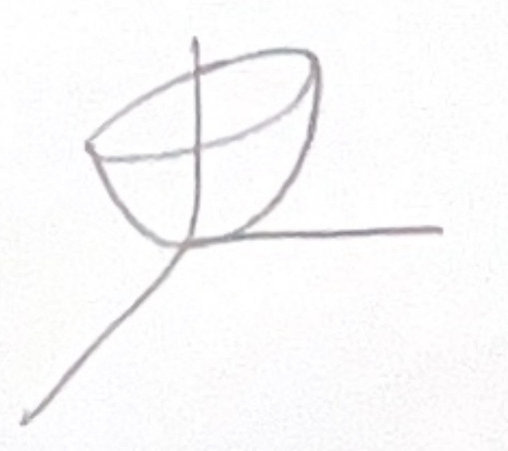


Hyperboloid of one sheet:

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

Elliptic Parabloid:

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$

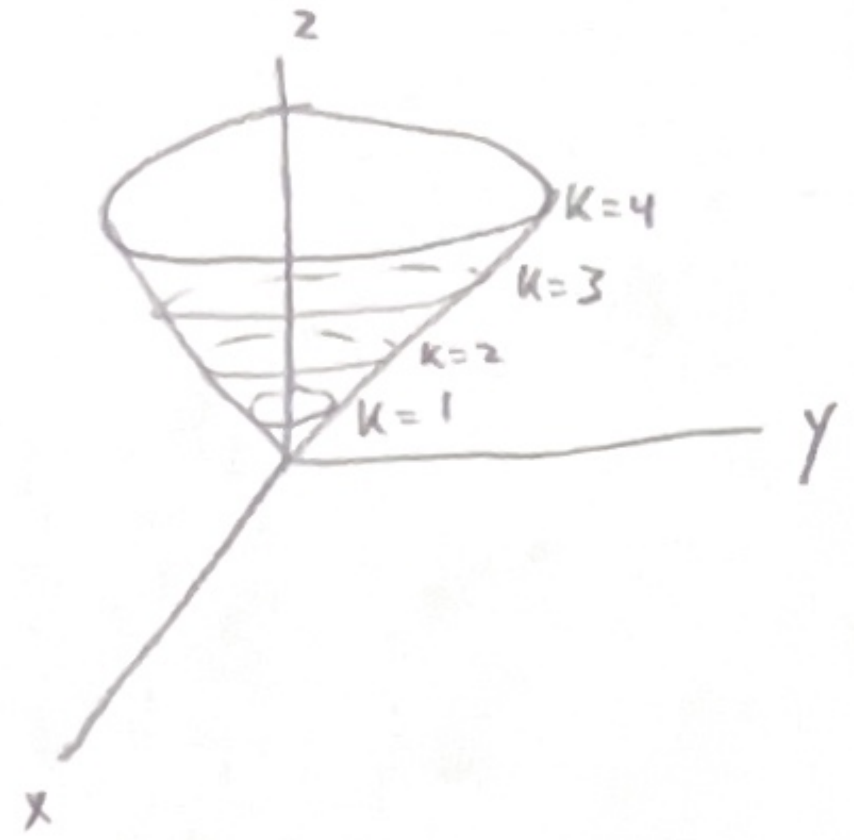


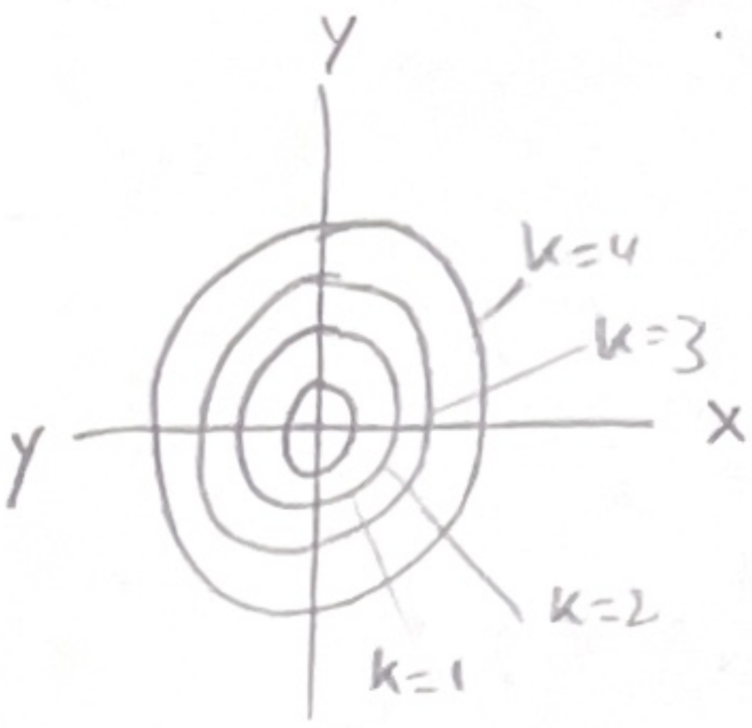
(upside down):

$z = -x^2 - y^2 + C$

Contour / Level Curves

$k = \sqrt{x^2+y^2} \rightarrow x^2+y^2=k^2$

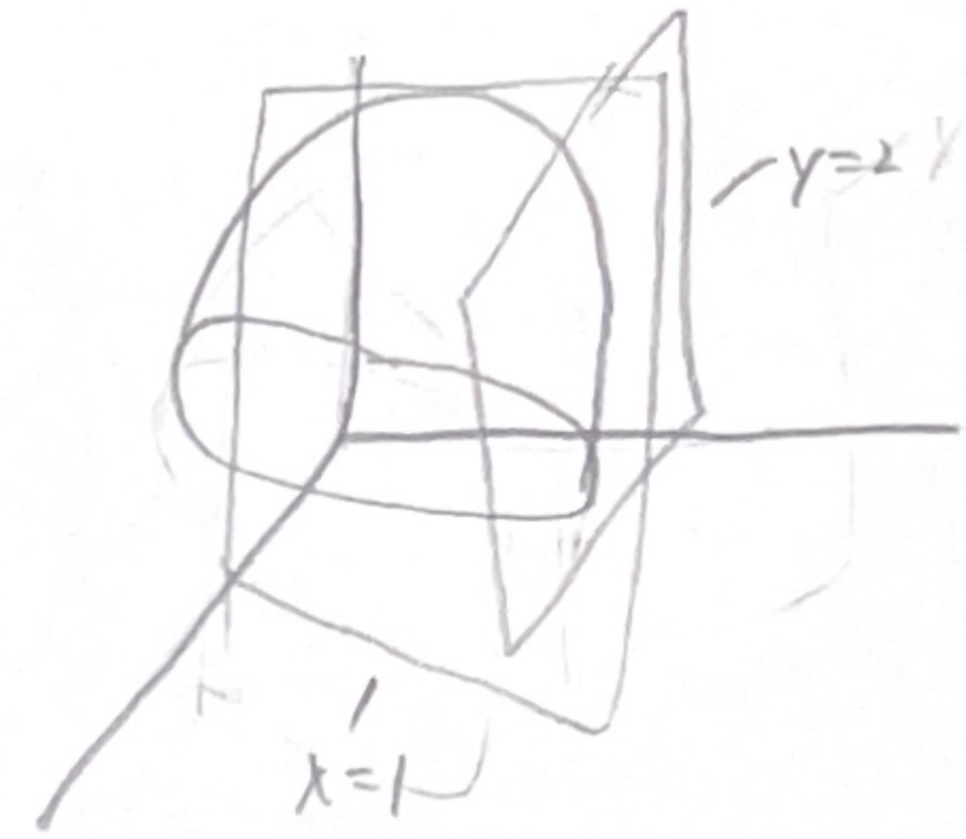




Traces

traces (Planes)

$f(x,y)=10-4x^2-y^2$, $x=1$, $y=2$



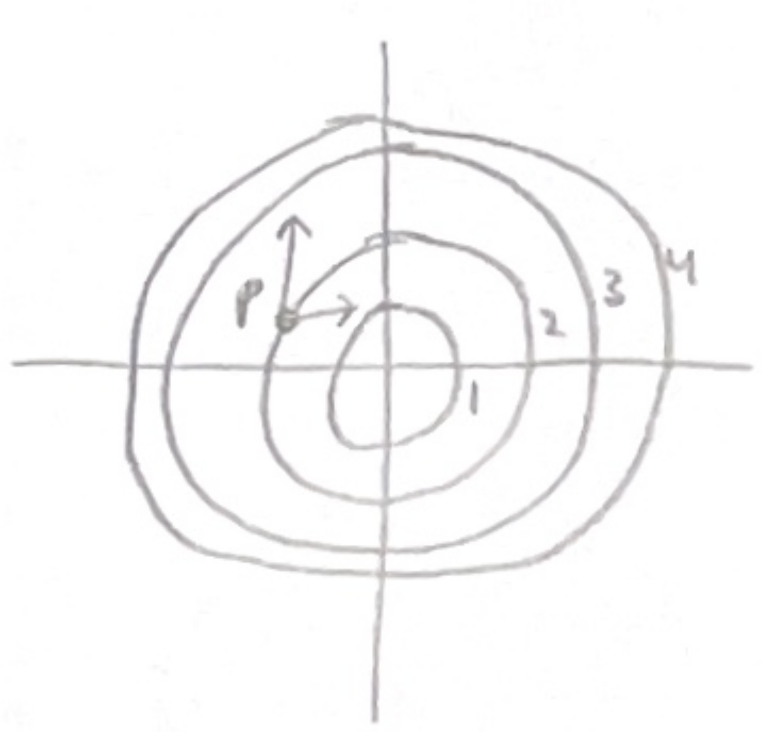
cut it with planes

Partial Derivatives

$f_x = \frac{\partial f}{\partial x}$ *treat y as a constant

$f_y = \frac{\partial f}{\partial y}$ *treat x as a constant

Positive or negative



f_x @ P = Negative

f_y @ P = Positive

2nd derivative

Case 1: First is ⊕ and levels are getting closer

Positive

Case 2: First is ⊕ and levels are getting farther apart

NEGATIVE

*flipped for negative

Chain rule (Implicit)

$\frac{dz}{dx}$ *differentiate x and z and attach a $\frac{dz}{dx}$ to the z's

Equation of tangent Plane

(at point $z=f(x,y)$ @ (x_0,y_0))

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

$* z_0=f(x_0,y_0)$

$$z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

Linear Approximation

$$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

$f(x,y) \approx L(x,y)$ near (x_0,y_0)

Directional Derivatives

$D_{\vec{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b$

where $\vec{u} = \langle a, b \rangle$

$D_{\vec{u}} f(x,y) = \langle f_x, f_y \rangle \cdot \langle a, b \rangle$

where $\langle f_x, f_y \rangle$ is the **gradient** of f

$\vec{\nabla} f = \langle f_x, f_y \rangle$

$\boxed{D_{\vec{u}} f = \nabla f \cdot \vec{u}}$

Relative max/mins

(a,b) is a critical point where $\nabla f(a,b) = \vec{0}$ or

$f_x = \text{DNE}$, $f_y = \text{DNE}$

$f_x = 0$, $f_y = 0$

$D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b)^2$

Cases:

1. $D > 0 \rightarrow$ max or min
   - $f_{xx} > 0 \rightarrow$ min
   - $f_{xx} < 0 \rightarrow$ max
2. $D < 0 \rightarrow$ Saddle point
3. $D = 0$, no information

Directional Derivative Max

max $= |\nabla f|$ and occurs in the direction given by ∇f

Lagrange Multipliers

f is optimized where

$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$

∇f is $\parallel$ to ∇g

So we solve:

$$\left.\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \\ g &= 0 \end{aligned}\right\} \text{4 eqn, 4 unknowns}$$

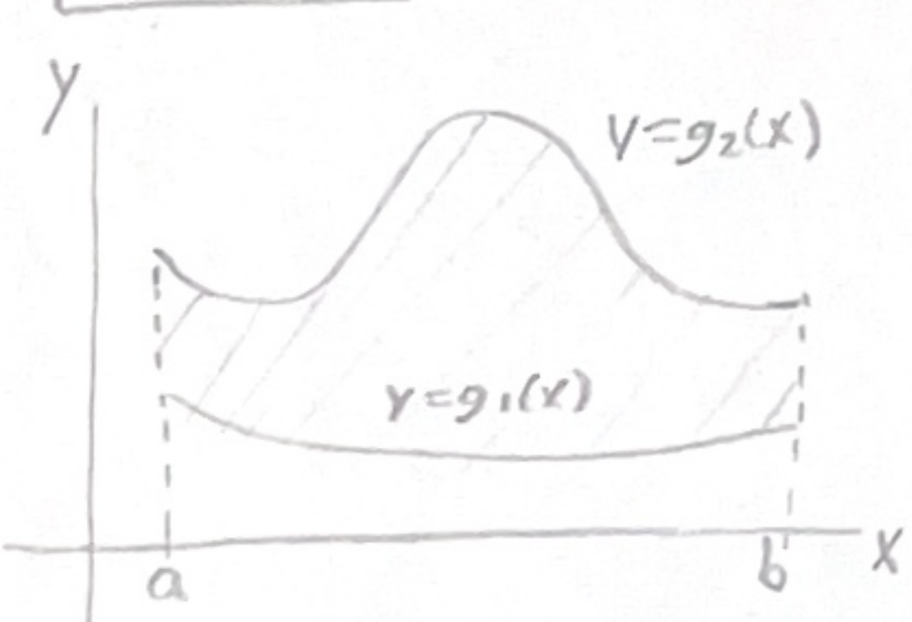
Things I need to Understand:

1. Implicit differentiation
2. Tangent planes
3. Linear approximations and Error

Multiple Integrals

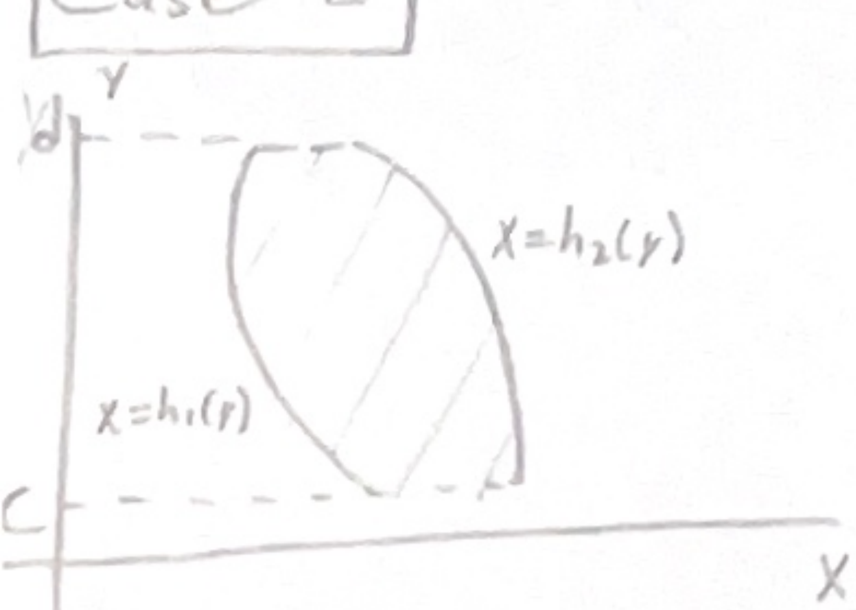
$$\text{Volume} = \iint_R f(x,y)\, dA$$

Case 1



$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

Case 2



$$\int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy$$

Double Integrals in Polar Coordinates

$$\int_{\alpha}^{\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta) \cdot r\, dr\, d\theta$$

$r^2 = x^2 + y^2$

$x = r\cos(\theta)$

$y = r\sin(\theta)$

Surface Area

Surface area of $z = f(x,y)$ above region D is given by

$$S = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA$$